Lecture 40

16.7 - Surface Integrals

Suppose we have a surface S which is parametrized by $\vec{r}(u,v)$, $(u,v) \in D$.

Recall from last Monday that

$$dS = |\vec{r}_u \times \vec{r}_v| \, dA$$

$\left(\vec{r}_u = \frac{\partial \vec{r}}{\partial u}, \ \vec{r}_v = \frac{\partial \vec{r}}{\partial v} \right)$.

As we did with curves, we can integrate functions along surfaces: $\iint_S f \, dS$.

Def: The surface integral of f over S (also called a scalar surface integral) is given by

$$\boxed{\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA}$$

(If $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ and $f = f(x,y,z)$, then
$f(\vec{r}(u,v)) = f(x(u,v), y(u,v), z(u,v))$.)

<u>Ex</u>: Compute the surface integral

$\iint_S xyz\, dS$ where S is the piece of the cone $z^2 = x^2 + y^2$ in the first octant, below $z=1$.

<u>Sol</u>: First, we parametrize the cone. It's easiest to use cylindrical coordinates here:

$$x = r\cos\theta,\ y = r\sin\theta,\ z = \sqrt{x^2+y^2} = r$$

(we take the positive root for z since we want the piece in the first octant, namely $z \geq 0$.)

So, $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$.

The domain of parametrization is: $0 \leq r \leq 1$,

$0 \leq \theta \leq \frac{\pi}{2}$ ← (this keeps x and y positive)

Now, we compute dS:

$\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$, $\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\,r.$$

So, $dS = \sqrt{2}\,r\,dA$.

Finally,

$$\iint_S xyz\,dS = \iint_D (r\cos\theta)(r\sin\theta)(r)(\sqrt{2}\,r)\,dA$$

$$= \sqrt{2}\int_0^1\int_0^{\pi/2} r^4\cos\theta\sin\theta\,d\theta\,dr$$

$$= \sqrt{2}\int_0^1 r^4\left(\tfrac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2} dr = \frac{\sqrt{2}}{2}\int_0^1 r^4\,dr$$

$$= \frac{\sqrt{2}}{10}$$

An Application

If we know the density function, $\rho(x,y,z)$, for a surface, we can find its mass and the coordinates of its center of mass, as before. The formulas are analogous to the line integral case:

$$\text{mass} = m = \iint_S \rho \, dS$$

center of mass $= (\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{1}{m}\iint_S x\rho \, dS, \quad \bar{y} = \frac{1}{m}\iint_S y\rho \, dS, \quad \bar{z} = \frac{1}{m}\iint_S z\rho \, dS$$

Ex: Find the mass of the hemisphere $x^2+y^2+z^2=25$, $z \le 0$, if it has density function $\rho(x,y,z) = \sqrt{2}\pi$.

Sol: First, we parametrize the hemisphere:

Use spherical coordinates: $\rho = 5$

$$\vec{r}(\theta, \varphi) = \langle 5\cos\theta\sin\varphi, 5\sin\theta\sin\varphi, 5\cos\varphi \rangle$$

$0 \le \theta \le 2\pi$, $\frac{\pi}{2} \le \varphi \le \pi$ (this makes $z \le 0$, i.e., below the xy-plane).

$$\vec{r}_\theta = \langle -5\sin\theta\sin\varphi, 5\cos\theta\sin\varphi, 0 \rangle$$

$$\vec{r}_\varphi = \langle 5\cos\theta\cos\varphi, 5\sin\theta\cos\varphi, -5\sin\varphi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \langle -25\cos\theta\sin^2\varphi, -25\sin\theta\sin^2\varphi, -25\cos^2\theta\sin\varphi\cos\varphi - 25\sin^2\theta\sin\varphi\cos\varphi \rangle$$

$$= -25\sin\varphi \langle \cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi \rangle$$

$$|\vec{r}_\theta \times \vec{r}_\varphi| = |-25\sin\varphi|\sqrt{\cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi}$$

$$= 25\sin\varphi \quad (\sin\varphi \geq 0 \text{ for } \tfrac{\pi}{2} \leq \varphi \leq \pi).$$

So,

$$\text{mass} = \iint_S \rho\, dS = \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi} \sqrt{2\pi}\,(25\sin\varphi)\, d\theta\, d\varphi$$

$$= 50\pi\sqrt{2\pi}\int_{\frac{\pi}{2}}^{\pi} \sin\varphi\, d\varphi = 50\pi\sqrt{2\pi}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right)$$

$$= 50\pi\sqrt{2\pi}\Big(-(-1) - (0)\Big) = 50\pi\sqrt{2\pi} = \sqrt{5000\pi^3}$$

$\diamond$

There is a formula for scalar surface integrals when the surface is a graph: $z = g(x,y)$, $(x,y) \in D$.

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

Once again, I find memorizing these extra formulas pointless, since you can just use the previous method.